2.12 Laplace transforms

Laplace transform theorems

Definition[a]	$F(s)=\mathscr{L}\{f(t)\}=\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$	(2.514)	$\mathscr{L}\{\}$	Laplace transform
Convolution[b]	$F(s)\cdot G(s)=\mathscr{L}\left\{\int_0^\infty f(t-z)g(z)\,\mathrm{d}z\right\}$	(2.515)	$F(s)$ $G(s)$	$\mathscr{L}\{f(t)\}$ $\mathscr{L}\{g(t)\}$
	$=\mathscr{L}\{f(t)*g(t)\}$	(2.516)	$*$	convolution
Inverse[c]	$f(t)=\frac{1}{2\pi\mathbf{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{e}^{st}F(s)\,\mathrm{d}s$	(2.517)	γ	constant
	$=\sum$ residues (for $t>0$)	(2.518)		
Transform of derivative	$\mathscr{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\}=s^n\mathscr{L}\{f(t)\}-\sum_{r=0}^{n-1}s^{n-r-1}\left.\frac{\mathrm{d}^r f(t)}{\mathrm{d}t^r}\right\vert_{t=0}$	(2.519)	n	integer >0
Derivative of transform	$\frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}=\mathscr{L}\{(-t)^n f(t)\}$	(2.520)		
Substitution	$F(s-a)=\mathscr{L}\{\mathrm{e}^{at}f(t)\}$	(2.521)	a	constant
Translation	$\mathrm{e}^{-as}F(s)=\mathscr{L}\{u(t-a)f(t-a)\}$	(2.522)	$u(t)$	unit step function
	where $u(t)=\begin{cases}0 & (t<0)\\ 1 & (t>0)\end{cases}$	(2.523)		

[a] If $\vert\mathrm{e}^{-s_0t}f(t)\vert$ is finite for sufficiently large t, the Laplace transform exists for $s>s_0$.
[b] Also known as the "faltung (or folding) theorem."
[c] Also known as the "Bromwich integral." γ is chosen so that the singularities in $F(s)$ are left of the integral line.

Laplace transform pairs

$$f(t) \Longrightarrow F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \qquad (2.524)$$

$$\delta(t) \Longrightarrow 1 \qquad (2.525)$$

$$1 \Longrightarrow 1/s \qquad (s>0) \qquad (2.526)$$

$$t^n \Longrightarrow \frac{n!}{s^{n+1}} \qquad (s>0, n>-1) \qquad (2.527)$$

$$t^{1/2} \Longrightarrow \sqrt{\frac{\pi}{4s^3}} \qquad (2.528)$$

$$t^{-1/2} \Longrightarrow \sqrt{\frac{\pi}{s}} \qquad (2.529)$$

$$\mathrm{e}^{at} \Longrightarrow \frac{1}{s-a} \qquad (s>a) \qquad (2.530)$$

$$t\mathrm{e}^{at} \Longrightarrow \frac{1}{(s-a)^2} \qquad (s>a) \qquad (2.531)$$

$$(1-at)\mathrm{e}^{-at} \Longrightarrow \frac{s}{(s+a)^2} \qquad (2.532)$$

$$t^2\mathrm{e}^{-at} \Longrightarrow \frac{2}{(s+a)^3} \qquad (2.533)$$

$$\sin at \Longrightarrow \frac{a}{s^2+a^2} \qquad (s>0) \qquad (2.534)$$

$$\cos at \Longrightarrow \frac{s}{s^2+a^2} \qquad (s>0) \qquad (2.535)$$

$$\sinh at \Longrightarrow \frac{a}{s^2-a^2} \qquad (s>a) \qquad (2.536)$$

$$\cosh at \Longrightarrow \frac{s}{s^2-a^2} \qquad (s>a) \qquad (2.537)$$

$$e^{-bt}\sin at \Longrightarrow \frac{a}{(s+b)^2+a^2} \qquad (2.538)$$

$$e^{-bt}\cos at \Longrightarrow \frac{s+b}{(s+b)^2+a^2} \qquad (2.539)$$

$$\mathrm{e}^{-at}f(t) \Longrightarrow F(s+a) \qquad (2.540)$$